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# Multidimensional Flux Difference Splitting Schemes

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**A novel alteration to the Cauchy–Kowalevski procedure is here presented to obtain essentially monotonic solutions for multidimensional flows. It is argued that this can be accomplished by splitting the cross-derivative terms among the several dimensions, such that the coefficient of the cross derivatives remains small compared to the coefficient of the normal derivatives. The approach naturally lends itself to extending the Roe flux difference splitting scheme to multiple dimensions and is advantaged over previous Cauchy–Kowalevski-based methods by yielding a solution free of spurious oscillations in the vicinity of oblique shock waves. Several test cases ranging from low-speed subsonic flows in channels to hypersonic flows over ramp injectors indicate that the proposed genuinely multidimensional method generally achieves a twofold or more increase in resolution along each dimension over the dimensionally split Roe scheme while retaining its appealing attributes: the scheme has a compact three-node-bandwidth stencil, is a finite volume flux function, yields essentially monotonic solutions, introduces minimal dissipation within viscous layers, and is written in general matrix form. Although the method proposed is first-order accurate, it offers a resolution as high or higher than the dimensionally split second-order total-variation-diminishing schemes for many problems of interest and is expected to surpass significantly the latter when extended to second-order accuracy.**



## Nomenclature

| | | |
|---|---|---|
| $A, B, C$ | = | flux Jacobian matrix along $x$, $y$, and $z$ |
| $a, b$ | = | wave speed along $x$ and $y$ |
| $C_f$ | = | skin-friction coefficient |
| $C_P$ | = | pressure coefficient |
| $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ | = | flux vector along $x$, $y$, and $z$ |
| $f, g$ | = | flux along $x$ and $y$ |
| $i, j, k$ | = | grid index along $x$, $y$, and $z$ |
| $L$ | = | left eigenvector matrix |
| $M$ | = | Mach number |
| $O$ | = | truncation error |
| $P$ | = | pressure |
| $p$ | = | order of accuracy |
| $R$ | = | right eigenvector matrix |
| $Re_x$ | = | Reynolds number along $x$ |
| $S$ | = | surface area of computational domain |
| $T$ | = | temperature |
| $t$ | = | time |
| $\boldsymbol{U}$ | = | vector of conserved variables |
| $u$ | = | conserved variable |
| $x, y, z$ | = | Cartesian coordinate |
| $\alpha$ | = | parameter related to obtaining essentially monotone solutions |
| $\beta$ | = | parameter related to the splitting of the second derivatives |
| $\gamma$ | = | ratio of the specific heats |
| $\Delta x, \Delta y, \Delta z$ | = | grid spacing along $x$, $y$, and $z$ |
| $\delta$ | = | entropy correction factor |
| $\epsilon$ | = | grid-induced error |
| $\Lambda$ | = | eigenvalue matrix |
| $\rho$ | = | density |

*Subscripts*

| | | |
|---|---|---|
| $c$ | = | coarse mesh |
| $f$ | = | fine mesh |
| $\infty$ | = | freestream conditions |

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## I. Introduction

MULTIDIMENSIONAL differential equations are commonly discretized by splitting the derivatives along each dimension and discretizing the so-obtained one-dimensional derivatives using one-dimensional operators. Referred to as "dimensional splitting," such a strategy suffers from being particularly dissipative when the grid is misaligned with the waves. This entails excessive grid refinement, and hence excessive computational effort, to correctly capture flows with waves propagating oblique to the grid lines. To remedy this problem, several genuinely multidimensional alternatives (i.e., multidimensional schemes that do not resort to dimensional splitting) have thus been proposed.

One approach that has been successful at reducing the dissipation of dimensional splitting is the rotation-interpolation method. Instead of calculating the fluxes in a coordinate system that is aligned with the grid, as is done with dimensional splitting, the fluxes are determined in a coordinate system that is rotated with respect to the grid through an interpolation of the node properties [1]. This can be extended to systems of conservation laws by solving a Riemann problem at the interfaces of the rotated cell [2–4]. The rotated Riemann solver approach has some advantages over dimensional splitting. For instance, in [2], it is shown that a first-order-accurate rotated Riemann solver achieves a resolution approaching the one of a higher-order MUSCL scheme when solving expansion fans and shocks. Further, in [3], it is demonstrated that a rotated Roe scheme has an advantage over its dimensionally split analog by being free of the aphysical carbuncle phenomenon. However, because the rotational frame is the same for all variables (the frame is typically rotated following the flow velocity), the rotated Riemann solver cannot capture all waves with a high resolution. For instance, the results obtained in [4] show that substituting a dimensionally split method by a rotated Riemann solver can be a mixed blessing: although it does improve the resolution of shock waves, it results in a poorer resolution of nonaligned shear waves.

Another approach that can be used to extend the Riemann solver to multiple dimensions is "residual distribution," as first proposed by Roe [5] and later improved by Deconinck et al. [6], Abgrall and Mezine [7], and Abgrall [8]. The residual distribution scheme treats the Riemann problem at the cell's interface in a genuinely multidimensional manner. This is accomplished by distributing the flux integral at the cells interfaces (the residual) to the neighboring nodes in a downwind manner, with the direction of downwinding being function of the waves within the Riemann solver. This yields advantages over dimensionally split methods: much less dissipation is introduced both at low and high Mach numbers (see, for instance, [9,10]), and the stencil is more compact for the same level of