2010 Heat Transfer Midterm Exam (Including Solutions)

EQUATE I AND II:

$$\frac{R_{ELECT}\, I^2 L}{\pi r_i^2} = \frac{2\pi L\,(T_i - T_\infty)}{\frac{\ln(r_o/r_i)}{k_{POLYETHYLENE}} + \frac{1}{h r_o}}$$

REWRITE AS:

$$r_i^2 = \left(\frac{\ln(r_o/r_i)}{k_{POLYETHYLENE}} + \frac{1}{h r_o}\right)\frac{R_{ELECT}\, I^2}{2\pi^2 (T_i - T_\infty)}$$

EXPRESS r_o AS $r_i + 0.005m$:

$$\boxed{r_i = \left[\left(\frac{\ln((r_i + 0.005m)/r_i)}{k_{POLYETHYLENE}} + \frac{1}{h(r_i + 0.005m)}\right)\frac{R_{ELECT}\, I^2}{2\pi^2 (T_i - T_\infty)}\right]^{1/2}} \quad \text{III}$$

TO ENSURE THAT THE TEMPERATURE WITHIN THE INSULATOR DOES NOT APPROACH ITS MELTING POINT BY LESS THAN 40°C, IT IS NECESSARY TO SET

$$T_i = 80°C$$

$$h = 7\ W/m^2°C$$

THEN ALL TERMS WITHIN EQ. III ARE KNOWN EXCEPT FOR r_i.

THEN DETERMINE r_i THROUGH PICARD ITERATIONS BY FIRST REWRITING EQ. III AS:

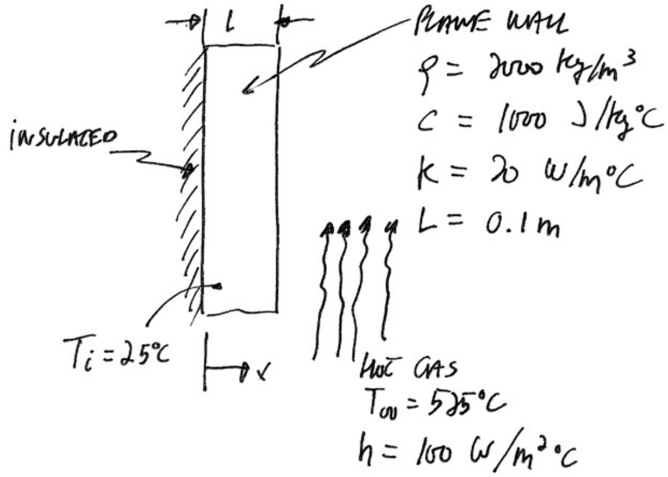
$$r_i^{n+1} = \left[\left(\frac{\ln((r_i^n + 0.005\,m)/r_i^n)}{0.5\ W/m°C} + \frac{1}{7\ W/m^2°C \times (r_i^n + 0.005\,m)} \right) \times 2.503 \times 10^{-6}\ \frac{Wm}{°C} \right]^{1/2}$$

n	r_i^n (m)	r_i^{n+1} (m)
1	0.005	0.0063
2	0.0063	0.0059
3	0.0059	0.0060
4	0.006	0.006

THEN, IT FOLLOWS THAT THE OPTIMAL RADIUS OF THE COPPER CABLE (RESULTING IN MINIMAL WEIGHT WHILE MAINTAINING THE POLYETHYLENE COATING TO A SAFE TEMPERATURE) IS OF:

$$\boxed{r_i = 0.006\ m}$$ (A)

#3 CONSIDER A PLANE WALL TO BE USED AS A THERMAL STORAGE UNIT:



WANTED:

- TIME t AT WHICH THE WALL HAS ACHIEVED 80% OF ITS MAXIMUM POSSIBLE ENERGY STORAGE
- FOR TIME t CALCULATED IN FIRST PART, FIND MAX. TEMP. IN WALL.

ASSUMPTIONS:

- CONSTANT ρ, c, k WITHIN WALL
- 1D HEAT TRANSFER WITHIN WALL
- CONSTANT h AND T_∞ OVER EXPOSED SURFACE
- NO RADIATION

SOLUTION:

FIRST, CALCULATE BIOT NUMBER:

$$Bi = \frac{hL}{k} = \frac{100\ W/m^2°C \times 0.1\ m}{20\ W/m°C} = 0.5$$

BECAUSE $Bi > 0.1$, CANNOT USE L.C.A. RESORT TO HEISLER CHARTS. THEN, CALCULATE Q/Q_0:

$$\frac{Q}{Q_0} = \frac{\rho V c\,(T_i - \bar{T})}{\rho V c\,(T_i - T_\infty)} = \frac{T_i - \bar{T}}{T_i - T_\infty}$$

WHEN WALL ACHIEVES 80% OF ITS MAXIMUM POSSIBLE ENERGY STORAGE; $\bar{T}$ WILL BE EQUAL TO:

$$\bar{T} = 0.80 \times (T_\infty - T_i) + T_i = 0.8\,T_\infty + 0.2\,T_i$$

SUBSTITUTE LATTER IN FORMER:

$$\frac{Q}{Q_0} = \frac{T_i - 0.8\,T_\infty - 0.2\,T_i}{T_i - T_\infty} = 0.8 \times \frac{(T_i - T_\infty)}{T_i - T_\infty} = 0.8$$

NOW, KNOWING $Q/Q_0 = 0.8$ AND $Bi = 0.5$, USE HEISLER CHART "INTERNAL ENERGY CHANGE AS A FUNCTION OF TIME FOR A PLANE WALL...." AND FIND $Bi^2 Fo$:

$$Bi^2 Fo = 1.0$$

THEN FROM $Bi^2 Fo = 1$, FIND t:

$$Bi^2 Fo = \frac{h^2 \alpha t}{k^2} = \frac{h^2 \cancel{k} t}{\rho c k^{\cancel{2}}} = \frac{h^2 t}{\rho c k}$$

ISOLATE t:

$$t = \frac{\rho c k (Bi^2 Fo)}{h^2} = \frac{2000 \text{ kg/m}^3 \times 1000 \text{ J/kg°C} \times 20 \text{ W/m°C} \times 1.0}{(100 \text{ W/m}^2\text{°C})^2}$$

$$\boxed{t = 4000 \text{ s}}$$ (14)

NOW, AT $t = 4000$ s, FIND MAX. TEMPERATURE IN WALL, T_{MAX}. NOTE THAT T_{MAX} IS LOCATED AT $x = L$, BECAUSE THIS IS THE CLOSEST POINT TO THE HOT ~~gas~~ GAS HEATING THE WALL. FIRST, FIND FOURIER #:

$$Fo = \frac{\alpha t}{L^2} = \frac{k t}{\rho c L^2} = \frac{20 \text{ W/m°C} \times 4000 \text{s}}{2000 \text{ kg/m}^3 \times 1000 \text{ J/kg°C} \times (0.1 \text{m})^2}$$

$$Fo = 4.0$$

THEN FIND T_{MAX} AS FOLLOWS:

$$\frac{T_{MAX} - T_\infty}{T_i - T_\infty} = \underbrace{\frac{T_0 - T_\infty}{T_i - T_\infty}}_{\substack{\text{USE CHART} \\ \text{"MIDPLANE TEMP."} \\ \text{WITH } Fo = 4.0 \\ \text{AND } Bi^{-1} = 2.0}} \times \underbrace{\frac{T_{MAX} - T_\infty}{T_0 - T_\infty}}_{\substack{\text{USE CHART "TEMP. DIST."} \\ \text{WITH } x/L = 1.0,\ Fo = 4.0 \\ \text{AND } Bi^{-1} = 2.0}}$$

THIS YIELDS

$$\frac{T_{MAX} - T_\infty}{T_i - T_\infty} = 0.2 \times 0.8 = 0.16$$

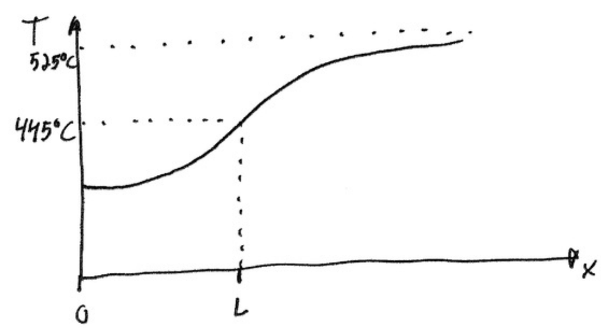
THEN ISOLATE T_{MAX}:

$$T_{MAX} = 0.16 \times (T_i - T_\infty) + T_\infty$$

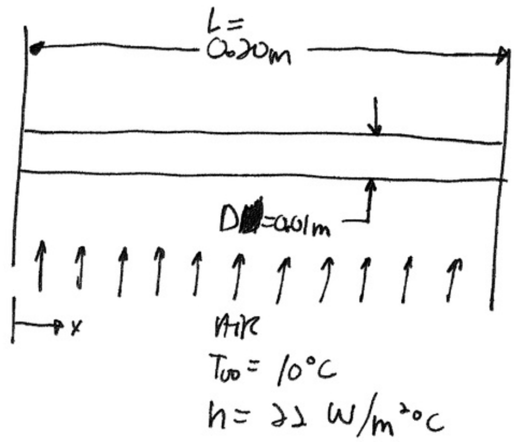
$$T_{MAX} = 0.16 \times (25°C - 525°C) + 525°C$$

$$\boxed{T_{MAX} = 445\ °C}$$ (A)

NOTE: THIS IS A HEATING PROBLEM, NOT A COOLING PROBLEM. THE TEMPERATURE DISTRIBUTION AT $t = 4000$ s IS HENCE AS FOLLOWS:



#4 CONSIDER HEAT TRANSFER IN A FIN JOINING TWO WALLS AT 110°C :



WANTED :

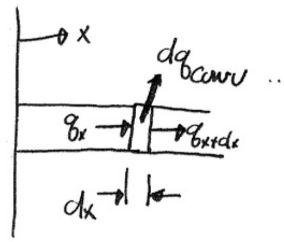
- FOR NO CONTACT RESISTANCE AT ROD SURFACE, T = 60°C AT x = 0.10m. FIND k.
- KNOWING k, FIND CONTACT CONDUCTANCE IF T = 70°C AT x = 0.10 m.

ASSUMPTIONS :

- 1D HEAT CONDUCTION ALONG x
- CONSTANT k, ρ, c WITHIN ROD
- CONSTANT h AND T_∞ ACTING ON ROD SURFACE.
- NO CONTACT RESISTANCE BETWEEN ROD AND WALLS
- RADIATION IS NEGLIGIBLE.
- STEADY - STATE

SOLUTION:

FIRST, WORK OUT AN ANALYTICAL EXPRESSION YIELDING T AS A FUNCTION OF x:



AT STEADY STATE, THE INTEGRAL FORM OF THE HEAT EQUATION YIELDS:

$$q_x = q_{x+dx} + dq_{conv}$$

WITH:

$$q_x = -k \frac{dT}{dx} \frac{\pi D^2}{4}$$

$$q_{x+dx} = -k \frac{dT}{dx} \frac{\pi D^2}{4} - \frac{d}{dx}\left(\frac{k \pi D^2}{4} \frac{dT}{dx} \right) dx + \ldots$$

SUBSTITUTE LATTER IN HEAT EQUATION:

$$\cancel{-\frac{\pi D^2 k}{4} \frac{dT}{dx}} - dq_{conv} = \cancel{-\frac{k \pi D^2}{4} \frac{dT}{dx}} - \frac{d}{dx}\left(\frac{k \pi D^2}{4} \frac{dT}{dx} \right) dx$$

FOR CONSTANT k, THIS YIELDS:

$$dq_{conv} = +\frac{k\pi D^2}{4}\frac{d^2T}{dx^2}dx$$

NOW USE RESISTANCE ANALOGY TO EXPRESS dq_{conv}:

$$dq_{conv} = \frac{-\Delta T}{\Sigma R} = \frac{-(T_\infty - T)}{\frac{1}{h\pi D dx} + \frac{1}{h_c \pi D dx}}$$

WITH h_c THE CONTACT CONDUCTANCE. REARRANGE:

$$dq_{conv} = \frac{(T - T_\infty)\pi D dx}{\frac{1}{h} + \frac{1}{h_c}}$$

SUBSTITUTE IN HEAT EQUATION:

$$\frac{\cancel{\pi}\cancel{D}\cancel{dx}\,(T - T_\infty)}{\frac{1}{h} + \frac{1}{h_c}} = +\frac{k\cancel{\pi}D^{\cancel{2}}}{4}\frac{d^2T}{dx^2}\cancel{dx}$$

OR

$$T - T_\infty = +\left(\frac{1}{h} + \frac{1}{h_c}\right)\frac{kD}{4}\frac{d^2T}{dx^2}$$

NOW DEFINE

$$\theta \equiv T - T_\infty$$

$$m^2 \equiv \frac{4}{kD\left(\frac{1}{h} + \frac{1}{h_c}\right)}$$

THEN THE HEAT EQ. BECOMES:

$$\frac{d^2\theta}{dx^2} = \theta m^2$$

FIND SOLUTION USING H-T TABLES:

$$\boxed{\theta = A\sinh(mx) + B\cosh(mx)}$$

APPLY B.C.s TO FIND A AND B; AT $x=0$,

$T = T_w$; THIS YIELDS:

$$\boxed{T_w - T_\infty = B}$$

ALSO, DUE TO SYMMETRY, AT $x = L/2$, $d\theta/dx = 0$:

$$\left.\frac{d\theta}{dx}\right|_{x=L/2} = 0 = Am\cosh\left(\frac{mL}{2}\right) + Bm\sinh\left(\frac{mL}{2}\right)$$

ISOLATE A:

$$\boxed{A = -B\tanh\left(\frac{mL}{2}\right)}$$

SUBSTITUTE A AND B IN GENERAL SOLUTION:

$$\theta = -B \tanh\left(\frac{mL}{2}\right)\sinh(mx) + B\cosh(mx)$$

SUBSTITUTE $\theta = T - T_\infty$ AND $B = T_w - T_\infty$:

$$T - T_\infty = -(T_w - T_\infty)\tanh\left(\frac{mL}{2}\right)\sinh(mx) + (T_w - T_\infty)\cosh(mx)$$

NOW OBTAIN SOLUTION FOR T_m AT $x = L/2$:

$$\frac{T_m - T_\infty}{T_w - T_\infty} = \cosh\left(\frac{mL}{2}\right) - \tanh\left(\frac{mL}{2}\right)\sinh\left(\frac{mL}{2}\right)$$

MULTIPLY BY $\cosh\left(\frac{mL}{2}\right)$ BOTH SIDES:

$$\left(\frac{T_m - T_\infty}{T_w - T_\infty}\right)\cosh\left(\frac{mL}{2}\right) = \cosh^2\left(\frac{mL}{2}\right) - \sinh^2\left(\frac{mL}{2}\right)$$

BUT $\sinh^2(x) - \cosh^2(x) = -1$:

$$\frac{T_m - T_\infty}{T_w - T_\infty} = \frac{1}{\cosh\left(\frac{mL}{2}\right)}$$

OR

$$\frac{mL}{2} = \cosh^{-1}\left(\frac{T_w - T_\infty}{T_m - T_\infty}\right)$$

BUT RECALL THAT

$$m = \sqrt{\frac{4}{kD\left(\frac{1}{h}+\frac{1}{h_c}\right)}}$$

SUBSTITUTE IN FORMER:

$$\frac{\cancel{4}}{kD\left(\frac{1}{h}+\frac{1}{h_c}\right)} = \frac{\cancel{4}}{L^2}\left[\cosh^{-1}\left(\frac{T_w - T_\infty}{T_m - T_\infty}\right)\right]^2$$

OR:

$$\boxed{k\left(\frac{1}{h}+\frac{1}{h_c}\right) = \frac{L^2}{D}\left[\cosh^{-1}\left(\frac{T_w - T_\infty}{T_m - T_\infty}\right)\right]^{-2}} \quad I$$

ISOLATE k IN EQ. I:

$$k = \frac{L^2}{D\left(\frac{1}{h}+\frac{1}{h_c}\right)}\left[\cosh^{-1}\left(\frac{T_w - T_\infty}{T_m - T_\infty}\right)\right]^{-2}$$

AND FOR NO CONTACT CONDUCTANCE ; $T_w = 110°C$, $T_m = 60°C$:

$$k = \frac{(0.20m)^2}{0.01m} \times \frac{22W}{m^2 °C} \times \left[\cosh^{-1}\left(\frac{110°C - 10°C}{60°C - 10°C}\right)\right]^{-2}$$

$$\boxed{k = 50.74 \frac{W}{m°C}}$$ A