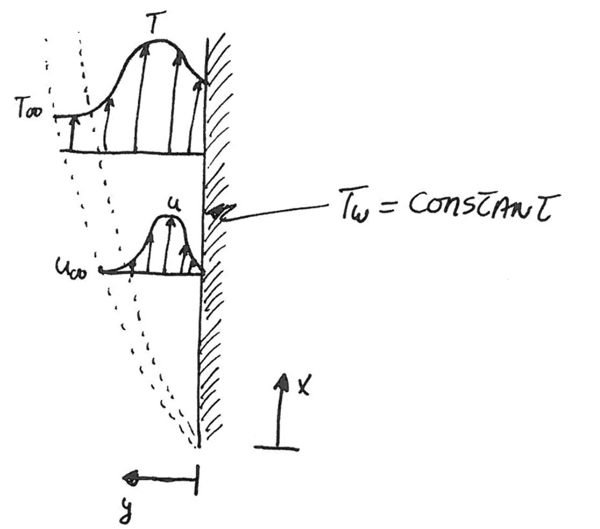
Heat Transfer Handout 2 — Natural Convection

BOUNDARY LAYER CREATED BY FREE CONVECTION:

(GO QUICKLY) (GIVE HANDOUT)

CONSIDER A THERMAL LAYER FORMED BY FREE CONVECTION OVER A VERTICAL FLAT PLATE:



FOR A GAS, NEGLECTING ALL SHEAR STRESSES EXCEPT THE ONE NORMAL TO THE WALL BUT INCLUDING THE BODY FORCE DUE TO GRAVITY THE x-MOMENTUM EQUATION BECOMES:

$$\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u v) = -\frac{\partial P}{\partial x} - \rho g + \mu \frac{\partial^2 u}{\partial y^2} \qquad \mathrm{I}$$

THE LATTER IS APPLICABLE AS WELL IN THE FREE STREAM:

$$\frac{\partial}{\partial x}\left(\rho_\infty u_\infty^2\right) + \frac{\partial}{\partial y}\left(\rho_\infty u_\infty v_\infty\right) = -\frac{\partial P_\infty}{\partial x} - \rho_\infty g + \mu \frac{\partial^2 u_\infty}{\partial y^2}$$

BUT $u_\infty = v_\infty = 0$. THEREFORE THE LATTER YIELDS:

$$0 = -\frac{\partial P_\infty}{\partial x} - \rho_\infty g$$

OR

$$\frac{\partial P_\infty}{\partial x} = -\rho_\infty g$$

UNDER THE BOUNDARY LAYER APPROXIMATION, IT IS REASONABLE TO ASSUME THAT THE PRESSURE DOES NOT VARY ALONG y:

$$\frac{\partial P}{\partial y} \approx 0$$

IF SO THEN

$$\frac{\partial P}{\partial x} = \frac{\partial P_\infty}{\partial x}$$

OR

$$\frac{\partial P}{\partial x} = -\rho_\infty g$$

THEN SUBSTITUTE THE LATTER IN THE x-MOMENTUM EQUATION (EQ. I):

$$\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u v) = \rho_\infty g - \rho g + \mu \frac{\partial^2 u}{\partial y^2}$$

BUT WE CAN REWRITE $\rho_\infty - \rho$ USING THE EQ. OF STATE:

$$\rho_\infty - \rho = \rho_\infty \left(1 - \frac{\rho}{\rho_\infty}\right)$$

$$= \rho_\infty \left(1 - \frac{P/RT}{P_\infty / RT_\infty}\right)$$

BUT SINCE $P \simeq P_\infty$ THROUGH THE BOUNDARY LAYER:

$$\rho_\infty - \rho = \rho_\infty \left(1 - \frac{T_\infty}{T}\right)$$

$$= \rho_\infty \left(\frac{T - T_\infty}{T}\right)$$

THEN, SUBSTITUTE THE LATTER IN THE MOMENTUM EQUATION:

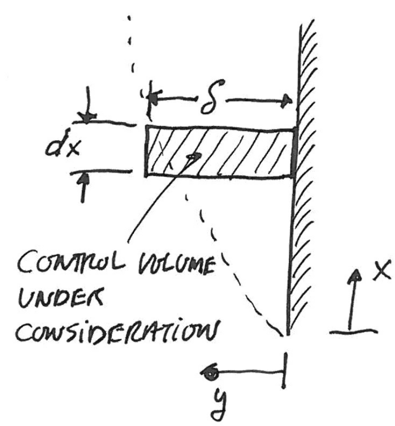
$$\boxed{\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u v) = \rho_\infty \beta g (T - T_\infty) + \mu \frac{\partial^2 u}{\partial y^2}} \quad \text{II}$$

WHERE β IS DEFINED AS:

$$\beta \equiv \frac{1}{T}$$

WHERE ABSOLUTE UNITS (e.g. KELVIN) MUST BE USED FOR THE TEMPERATURE SINCE THE LATTER WAS DERIVED WITH THE HELP OF THE EQUATION OF STATE ($P = \rho R T$) IN WHICH T HAS ABSOLUTE UNITS.

THEN, FOLLOWING A SIMILAR STRATEGY AS USED PREVIOUSLY WHEN DERIVING THE NUSSELT NUMBER FOR EXTERNAL AND INTERNAL FORCED CONVECTION, LET'S INTEGRATE THE MOMENTUM EQUATION OVER A C.V. OF HEIGHT δ AND LENGTH dx:



THEN, MULTIPLY THE MOMENTUM EQ BY dy AND INTEGRATE FROM $y=0$ to $y=\delta$:

$$\int_0^\delta \frac{\partial}{\partial x}(\rho u^2)\,dy + \int_0^\delta d(\rho u v) = \int_0^\delta \rho_\infty \beta g (T-T_\infty)\,dy + \int_0^\delta \mu\, d\left(\frac{\partial u}{\partial y}\right)$$

LET'S INTEGRATE EACH TERM SEPARATELY; THE 1ST ONE:

$$\int_0^\delta \frac{\partial}{\partial x}(\rho u^2)\,dy = \frac{d}{dx}\int_0^\delta \rho u^2\,dy$$

SINCE THE INTEGRAL OF A DERIVATIVE IS THE DERIVATIVE OF THE INTEGRAL.

THE 2nd ONE COLLAPSES TO ZERO:

$$\int_0^\delta d(\rho u v) = 0$$

SINCE $u_{y=\delta} = v_{y=\delta} = 0$ AND $u_{y=0} = v_{y=0} = 0$.

AND THE LAST ONE ON THE RHS BECOMES:

$$\int_0^\delta \mu\, d\left(\frac{\partial u}{\partial y}\right) = -\mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

SINCE $(\partial u/\partial y)_{y=\delta} = 0$.

SUBSTITUTING THE LATTER BACK IN THE MOMENTUM EQUATION, WE GET:

$$\frac{d}{dx}\int_0^\delta \rho u^2\,dy = -\mu \left.\frac{\partial u}{\partial y}\right|_{y=0} + \int_0^\delta \rho_\infty g \beta (T-T_\infty)\,dy$$

BUT, SINCE ρ CHANGES BY A RELATIVELY SMALL AMOUNT COMPARED TO u^2 WE CAN SAY:

$$\frac{d}{dx}\int_0^\delta \rho u^2\,dy \approx \rho_\infty \frac{d}{dx}\int_0^\delta u^2\,dy$$

SUBSTITUTING THE LATTER IN THE FORMER AND DIVIDING THROUGH BY ρ_∞ YIELDS:

$$\boxed{\frac{d}{dx}\int_0^\delta u^2\,dy = -\nu \left.\frac{\partial u}{\partial y}\right|_{y=0} + \int_0^\delta g \beta (T-T_\infty)\,dy} \quad \text{III}$$

WHERE ν IS THE KINEMATIC VISCOSITY DEFINED AS μ/ρ_∞. NOW, TO INTEGRATE THE LATTER WE NEED TO FIND u AND T AS A FUNCTION OF y.

TO FIND T AS A FUNCTION OF y, ASSUME T IS A SECOND DEGREE POLYNOMIAL OF THE FORM:

$$T = a + by + cy^2$$

WHERE THE COEFFICIENTS a, b, c ARE FOUND FROM THE BOUNDARY CONDITIONS:

(i) AT $y=0$, $T=T_w$

(ii) AT $y=\delta_t$, $T=T_\infty$

(iii) AT $y=\delta_t$, $\dfrac{\partial T}{\partial y}=0$

ASSUMES THAT $\delta \simeq \delta_t$

THEN IT CAN BE SHOWN THAT

$$\boxed{\frac{T-T_\infty}{T_w-T_\infty} = \left(1-\frac{y}{\delta_t}\right)^2} \qquad \text{IV}$$

SIMILARLY, WE CAN APPROXIMATE THE VELOCITY PROFILE THROUGH A POLYNOMIAL FIT AT THE BOUNDARIES. ASSUME IN THIS CASE A THIRD DEGREE POLYNOMIAL:

$$u = a + by + cy^2 + dy^3$$

WHERE THE COEFFICIENTS a, b, c, d ARE FOUND FROM THE FOLLOWING 4 BDRY CONDITIONS:

(i) AT $y=0$, $u=0$

(ii) AT $y=\delta$, $u=0$

(iii) AT $y=\delta$, $\dfrac{\partial u}{\partial y}=0$

AND THE LAST B.C. CORRESPONDS TO:

$$(iv) \text{ AT } y=0, \quad \frac{\partial^2 u}{\partial y^2} = -g\beta \frac{T_w - T_\infty}{\nu}$$

THE LATTER COMES FROM THE MOM. EQ. (EQ. II) EVALUATED AT $u=0$, $v=0$; $T=T_\infty$:

$$\cancelto{0}{\frac{\partial}{\partial x}(\rho u^2)} + \cancelto{0}{\frac{\partial}{\partial y}(\rho u v)} = \rho_\infty \beta g (T - T_\infty) + \mu \frac{\partial^2 u}{\partial y^2}$$

AFTER APPLYING THE 4 B.C.s, THE POLYNOMIAL APPROXIMATING THE VELOCITY BECOMES:

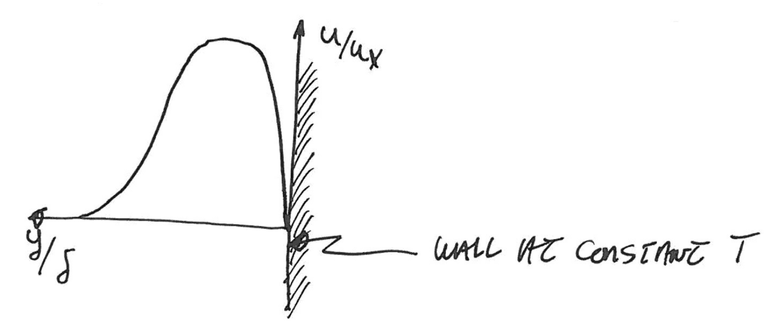
$$\boxed{u = u_x \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2} \qquad \text{V}$$

WITH u_x DEFINED AS

$$u_x \equiv \frac{\beta \delta^2 g (T_w - T_\infty)}{4\nu}$$

WITH ν THE KINEMATIC VISCOSITY $\nu \equiv \mu / \rho_\infty$.

SINCE u_x IS ONLY A FUNCTION OF x, WE CAN READILY PLOT THE VELOCITY PROFILE AS A FUNCTION OF y/δ:



THEN, SUBSTITUTING THE VELOCITY OBTAINED IN EQ. V AND THE TEMPERATURE OBTAINED IN EQ. IV IN THE MOMENTUM EQUATION (EQ. III) AND INTEGRATING, WE GET:

$$\frac{1}{105}\frac{d}{dx}\left(u_x^2\,\delta\right) = \frac{1}{3}\, g\beta\,(T_w - T_\infty)\,\delta - \frac{\nu u_x}{\delta} \qquad \mathrm{VI}$$

THEN IT CAN BE EASILY SHOWN THAT THE ENERGY EQUATION CAN BE WRITTEN AS: (NO VISCOUS DISSIPATION)

$$\frac{d}{dx}\left(\int_0^\delta u\,(T - T_\infty)\,dy\right) = -\alpha \left.\frac{dT}{dy}\right|_{y=0}$$

SUBSTITUTE T FROM IV AND u FROM V IN THE LATTER AND INTEGRATE:

$$\frac{1}{30}(T_w - T_\infty)\frac{d}{dx}\left(u_x\,\delta\right) = 2\alpha\,\frac{(T_w - T_\infty)}{\delta} \qquad \mathrm{VII}$$

BUT RECALL THAT u_x WAS DEFINED AS

$$u_x \equiv \frac{\beta\,\delta^2\, g\,(T_w - T_\infty)}{4\nu}$$

AFTER SUBSTITUTING THE LATTER IN VI WE FIND

$$\delta \sim x^{1/4}$$

SUBSTITUTE THE LATTER IN FORMER:

$$u_x \sim x^{1/2}$$

THEN LET'S ASSUME THAT

$$u_x = C_1 x^{1/2}$$

$$\delta = C_2 x^{1/4}$$

SUBSTITUTE THE LATTER IN VI AND VII:

$$\frac{5}{420} C_1^2 C_2 x^{1/4} = g\beta(T_w - T_\infty)\frac{C_2}{3} x^{1/4} - \frac{C_1}{C_2} \nu x^{1/4}$$

$$\frac{1}{40} C_1 C_2 x^{-1/4} = \frac{2\alpha}{C_2} x^{-1/4}$$

THESE TWO EQUATIONS CAN YIELD C_1 AND C_2:

$$C_1 = 5.17\,\nu\left(\frac{20}{21} + \frac{\nu}{\alpha}\right)^{-1/2}\left(\frac{g\beta(T_w - T_\infty)}{\nu^2}\right)^{1/2}$$

$$C_2 = 3.93\left(\frac{20}{21} + \frac{\nu}{\alpha}\right)^{1/4}\left(\frac{g\beta(T_w - T_\infty)}{\nu^2}\right)^{-1/4}\left(\frac{\nu}{\alpha}\right)^{-1/2}$$

BUT RECALL

$$\frac{\delta}{x} = C_2 x^{-3/4} \quad (\text{FROM } \delta \sim C_2 x^{1/4})$$

AND NOTE THAT

$$\frac{\nu}{\alpha} = \frac{\mu}{\rho_\infty}\frac{\rho c_p}{k} \simeq \frac{\mu c_p}{k} = Pr$$

THEN:

$$\frac{\delta}{x} \simeq 3.93\left(\frac{20}{21}+Pr\right)^{1/4}\left(\frac{g\beta(T_w-T_\infty)x^3}{\nu^2}\right)^{-1/4}\left(Pr\right)^{-1/2}$$

DEFINE GRASHOF NUMBER:

$$Gr_x \equiv \frac{g\beta(T_w-T_\infty)x^3}{\nu^2}$$

THEN

$$\boxed{\frac{\delta}{x} \simeq 3.93\, Pr^{-1/2}(0.952+Pr)^{1/4}\, Gr_x^{-1/4}}$$

ASSUMPTIONS:

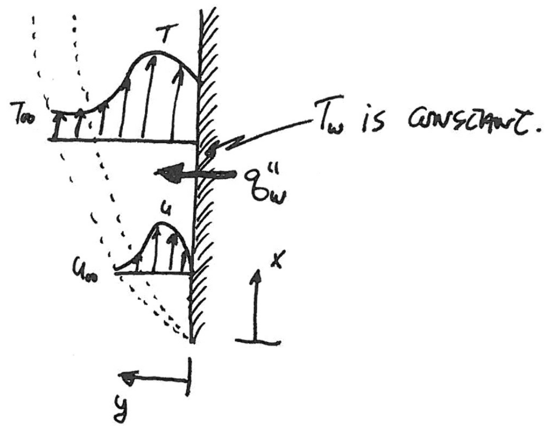
- $\delta = \delta_t$ } SIGNIFICANT ERROR FOR $Pr \gg 1$
- CONSTANT T_{WALL}
- NO VISCOUS DISSIPATION
- $u_x = C_1 x^{1/2} + \cancel{\ldots}$ neglect
- $\delta = C_2 x^{1/4} + \cancel{\ldots}$ neglect

  } SIGNIFICANT ERROR ??
- T is A 2nd DEGREE POLYNOMIAL } POSSIBLE LARGE ERROR
- u is A 3rd DEGREE POLYNOMIAL
- $\left|\frac{T-T_\infty}{T_\infty}\right| \ll 1$

FREE CONVECTION HEAT TRANSFER (VERTICAL PLATE)

(GO QUICKLY – GIVE HANDOUT)

RECALL THE THERMAL AND BOUNDARY LAYER FORMED BY FREE CONVECTION OVER A VERTICAL FLAT PLATE:



WE SHOWED THAT THE TEMPERATURE PROFILE CORRESPONDS APPROXIMATELY TO:

$$\frac{T-T_\infty}{T_w-T_\infty} = \left(1-\frac{y}{\delta}\right)^2 \qquad \mathrm{I}$$

AND WE SHOWED THAT

$$\frac{\delta}{x} = 3.93\, Pr^{-1/2}\,(0.952+Pr)^{1/4}\, Gr_x^{-1/4} \qquad \mathrm{II}$$

NOW, RECALL THAT THE HEAT FLUX AT THE WALL CORRESPONDS TO:

$$q''_w = -k \frac{\partial T}{\partial y}\bigg|_{y=0}$$

THEN FIND $\partial T/\partial y$ BY TAKING THE DERIVATIVE WITH RESPECT TO y ON BOTH SIDES OF EQ. [redacted] I:

$$\frac{\partial T}{\partial y} = 2(T_w - T_\infty)\left(1 - \frac{y}{\delta\text{[redacted]}}\right)\left(\frac{-1}{\delta\text{[redacted]}}\right)$$

AT $y=0$, THE LATTER YIELDS:

$$\frac{\partial T}{\partial y}\bigg|_{y=0} = -\frac{2(T_w - T_\infty)}{\delta\text{[redacted]}}$$

SUBSTITUTE THE LATTER IN THE EQ. FOR q''_w:

$$q''_w = \frac{2k}{\delta\text{[redacted]}}(T_w - T_\infty)$$

THEN RECALL THAT THE DEFINITION OF THE CONVECTIVE HEAT TRANSFER COEFFICIENT YIELDS:

$$q''_w = h(T_w - T_\infty)$$

COMPARING THE LATTER WITH THE FORMER IT IS EVIDENT THAT:

$$h = \frac{2k}{\delta\,\text{[redacted]}}$$

THEN RECALL THE DEFINITION OF THE NUSSELT NUMBER:

$$Nu_x \equiv \frac{h x}{k}$$

SUBSTITUTE THE FORMER IN THE LATTER:

$$Nu_x = \frac{2\cancel{k}}{\cancel{k}} \frac{x}{\delta\,\text{[redacted]}}$$

THEN SUBSTITUTE δ/x FROM EQ. II:

$$Nu_x = \frac{2}{3.93} Pr^{1/2} (0.952 + Pr)^{-1/4} Gr_x^{1/4}$$

OR

$$\boxed{Nu_x \approx 0.5\, Pr^{1/2} (0.952 + Pr)^{-1/4} Gr_x^{1/4}} \qquad \text{III}$$

NOW NOTE THE FOLLOWING:

IF $Pr \sim 1$ THEN $(0.952 + Pr)^{-1/4} \sim 1 \sim Pr^{-1/4}$

IF $Pr \gg 1$ THEN $(0.952 + Pr)^{-1/4} \sim Pr^{-1/4}$

THEREFORE WE CAN SAY:

$$(0.952 + Pr)^{-1/4} = Pr^{-1/4} \qquad \text{IV}$$

IF $Pr \gtrapprox 1$. FOR $Pr \sim 0.7$ THE LATTER ENTAILS AN ERROR OR 12% OR SO. SUCH AN ERROR IS SMALL COMPARED TO THE ERROR ORIGINATING FROM THE ASSUMPTIONS MADE IN DERIVING EQ. III. THEN SUBSTITUTE IV IN III:

$$Nu_x \approx 0.5\, Pr^{1/2}\, Pr^{-1/4}\, Gr_x^{1/4}$$

THIS YIELDS

$$\boxed{Nu_x \approx 0.5\, Pr^{1/4}\, Gr_x^{1/4}}$$

NOW DEFINE THE RAYLEIGH NUMBER AS:

$$Ra_x \equiv Gr_x \, Pr$$

THEN THE NUSSELT # BECOMES

$$\boxed{Nu_x = 0.5 \, Ra_x^{1/4}}$$

FURTHER, IT CAN BE EASILY SHOWN THAT THE AVERAGE NUSSELT # OVER THE LENGTH OF THE PLATE CORRESPONDS TO 4/3 THE LOCAL NUSSELT # AT $x=L$:

$$\overline{Nu_L} = \frac{4}{3} \times 0.5 \times Ra_L^{1/4}$$

OR

$$\boxed{\overline{Nu_L} = 0.67 \, Ra_L^{1/4}}$$

PHYSICAL SIGNIFICANCE OF RAYLEIGH #:

$$Ra = \frac{\text{BUOYANCY FORCES}}{\text{THERMAL} \times \text{MOMENTUM DIFFUSIVITY}}$$

$$Gr = \frac{\text{BUOYANCY FORCE}}{\text{VISCOUS FORCE}}$$

FREE CONVECTION H-T OVER COMPLEX GEOMETRIES

AS WAS SHOWN IN THE PREVIOUS SECTION, H-T OVER A VERTICAL FLAT PLATE DUE TO FREE CONVECTION CAN BE FOUND FROM

$$Nu_x = 0.5\, Ra_x^{1/4}$$

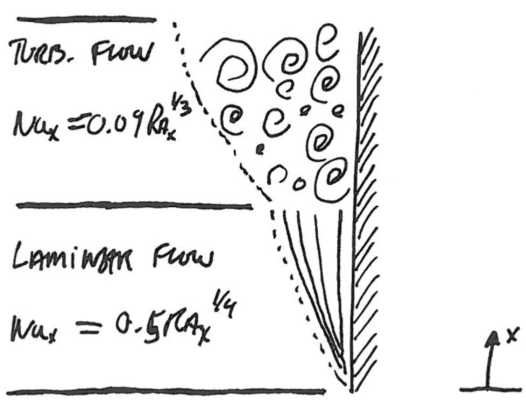
AND

$$\overline{Nu}_L = 0.67\, Ra_L^{1/4}$$

BUT THE LATTER IS ONLY APPLICABLE FOR LAMINAR FLOW OVER A FLAT PLATE.

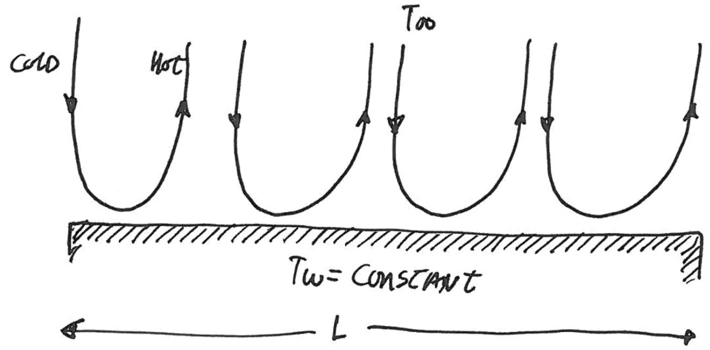
FOR TURBULENT FLOW, EXPERIMENTS INDICATE THAT:

$$Nu_x \simeq 0.09\, Ra_x^{1/3}$$



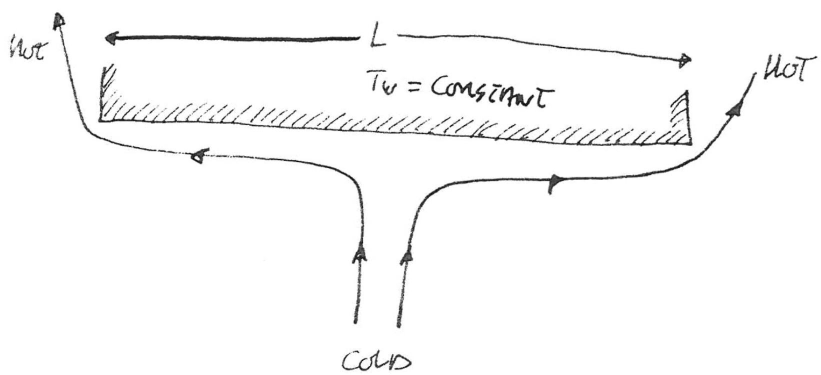
BUT, WHAT ABOUT OTHER GEOMETRIES?

UPPER SURFACE OF HEATED PLATE: ($T_w > T_\infty$)



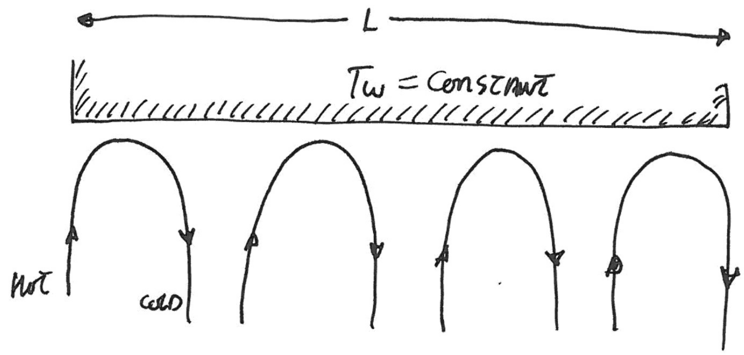
$$Nu_L = 0.15\,(Ra_L)^{1/3} \quad \text{FOR} \quad 8\times10^{6} < Ra_L < 10^{11}$$

LOWER SURFACE OF HEATED PLATE: ($T_w > T_\infty$)



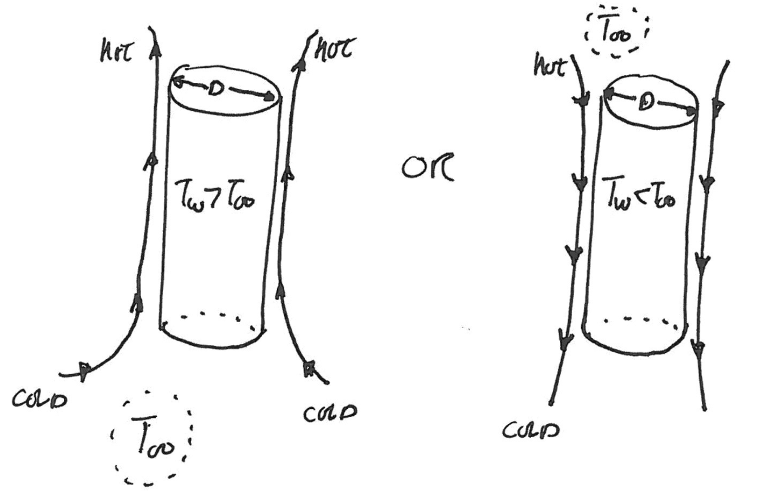
$$Nu_L = 0.27\,(Ra_L)^{1/4} \quad \text{FOR} \quad 10^{5} < Ra_L < 10^{11}$$

LOWER SURFACE OF COOLED PLATES: $(T_w < T_\infty)$



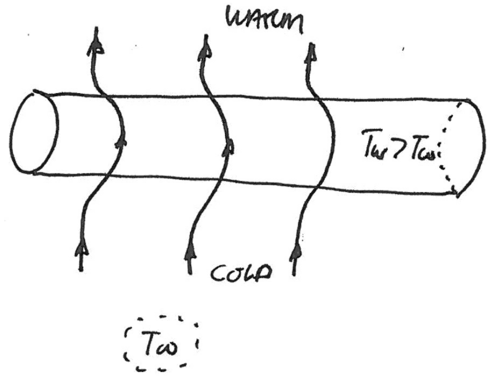
$$Nu_L = 0.15 (Ra_L)^{1/3} \quad \text{FOR} \quad 8 \times 10^6 < Ra_L < 10^{11}$$

VERTICAL CYLINDER $(T_w < T_\infty$ OR $T_w > T_\infty)$:

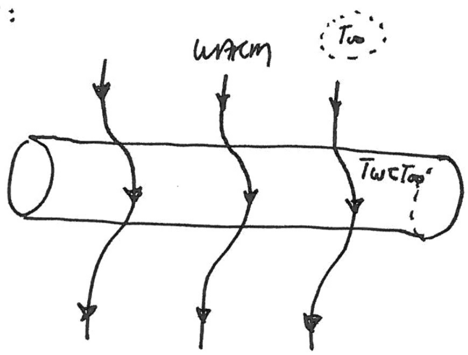


$$Nu_D = 0.775 (Ra_D)^{0.21} \quad \text{FOR} \quad 10^4 < Ra_D < 10^8$$

HORIZONTAL CYLINDERS ($T_w < T_\infty$ OR $T_w > T_\infty$)



OR:



THEN USE:

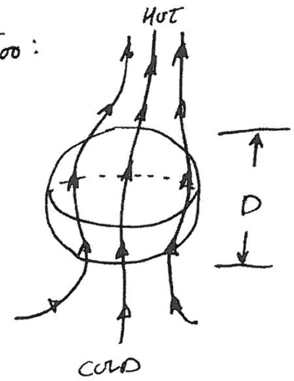
$$Nu_D^{0.5} = 0.6 + 0.387 \left[\frac{Ra_D}{\left[1 + (0.559/Pr)^{9/16} \right]^{16/9}} \right]^{1/6}$$

WIDE RANGE OF APPLICABILITY:

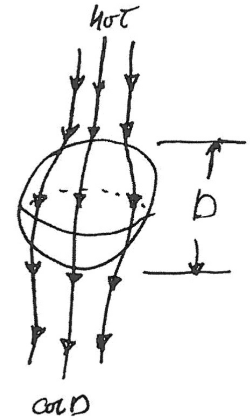
$$10^{-5} < Ra_D < 10^{12}$$

SPHERE ($T_w < T_\infty$ OR $T_w > T_\infty$)

$T_w > T_\infty$:



$T_w < T_\infty$:

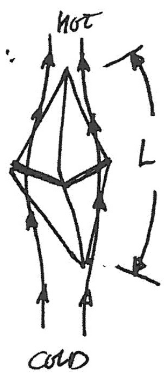


$$Nu_D = 2 + 0.43 (Ra_D)^{1/4} \quad \text{FOR} \quad 1 < Gr_D < 10^5$$

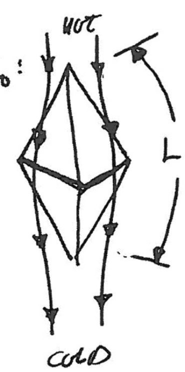
$$Nu_D = 2 + 0.5 (Ra_D)^{1/4} \quad \text{FOR} \quad 3\times10^5 < Ra_D < 8\times10^8$$

IRREGULAR SOLIDS ($T_w < T_\infty$ OR $T_w > T_\infty$)

$T_w > T_\infty$:



$T_w < T_\infty$:



$$Nu_L = 0.52 (Ra_L)^{1/4}$$

$L \equiv$ CHARACTERISTIC DISTANCE A FLUID PARTICLE TRAVELS IN BOUNDARY LAYER