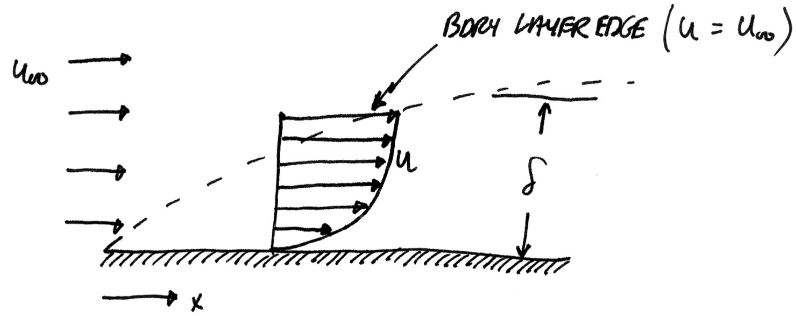
Heat Transfer Handout 1 — Boundary Layer

LAMINAR BOUNDARY LAYER – THICKNESS

CONSIDER A LAMINAR BOUNDARY LAYER OVER A FLAT PLATE:

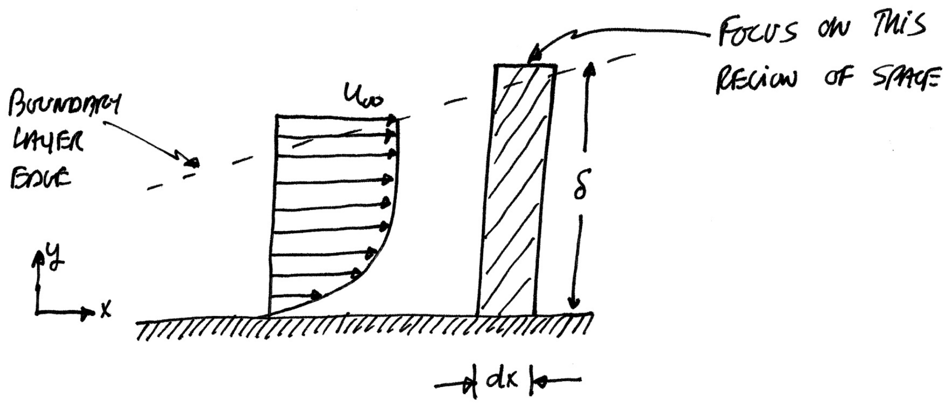


WANTED:

- δ AS A FUNCTION OF x

SOLUTION:

CONSIDER A REGION OF LENGTH dx IN WHICH THE BOUNDARY LAYER IS LOCATED:



RECALL THE X-MOMENTUM CONSERVATION EQUATION:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + P)}{\partial x} + \frac{\partial(\rho v u)}{\partial y} = \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2}$$

NOW, MAKE THE FOLLOWING ASSUMPTIONS:

- STEADY - STATE $\rightarrow \partial(\rho u)/\partial t = 0$
- NO PRESSURE GRADIENT $\rightarrow \partial P/\partial x = 0$
- NEGLIGIBLE SHEAR STRESS ALONG $x \rightarrow \mu\, \partial^2 u/\partial x^2 = 0$

NOW, THE X-MOMENTUM N-S EQ. BECOMES:

$$\frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho v u)}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2}$$

MULTIPLY BY $dx\, dy$ AND INTEGRATE OVER THE REGION UNDER CONSIDERATION:

$$\int_0^{\delta}\int_x^{x+dx} d(\rho u^2)\, dy + \int_x^{x+dx}\int_0^{\delta} d(\rho v u)\, dx = \int_x^{x+dx}\int_0^{\delta} \mu\, d\left(\frac{du}{dy}\right) dx$$

NOW, NOTE THAT:

- AT $y = 0$, $\rho v u = 0$ SINCE $v = 0$
- AT $y = \delta$, $\rho v u = 0$ SINCE $v = 0$
- AT $y = \delta$, $du/dy = 0$

THEN THE LATTER BECOMES:

$$\int_0^{\delta}\int_x^{x+dx} d(\rho u^2)\,dy = -\int_x^{x+dx} \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} dx$$

FURTHER NOTE THAT:

$$\rho u^2\Big|_{x+dx} = \rho u^2 + \frac{\partial(\rho u^2)}{\partial x}dx$$

AND THAT:

$$\mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \text{CONSTANT FROM } x \text{ to } x+dx$$

SINCE $dx \to 0$. THEN THE INTEGRAL BECOMES:

$$\int_0^{\delta}\left(\cancel{\rho u^2} + \frac{\partial(\rho u^2)}{\partial x}dx - \cancel{\rho u^2}\right)dy = -\mu \left.\frac{\partial u}{\partial y}\right|_{y=0} dx$$

DIVIDE THROUGH BY dx:

$$\int_0^{\delta} \frac{\partial(\rho u^2)}{\partial x}\,dy = -\mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

AND NOTE THAT THE INTEGRAL OF A DERIVATIVE IS EQUAL TO THE DERIVATIVE OF AN INTEGRAL:

$$\boxed{\frac{d}{dx}\int_0^{\delta} \rho u^2\,dy = -\mu \left.\frac{\partial u}{\partial y}\right|_{y=0}} \qquad \mathrm{I}$$

KEEP THE LATTER ON HOLD FOR A MOMENT.

PERFORM SIMILAR STEPS WITH THE MASS CONSERVATION EQUATION:

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

MULTIPLY THE LATTER BY $dx\,dy$ AND INTEGRATE OVER THE REGION UNDER CONSIDERATION:

$$\int_0^{\delta}\int_x^{x+dx} d(\rho u)\,dy + \int_x^{x+dx}\int_0^{\delta} d(\rho v)\,dx = 0$$

AND NOTE THAT

- AT $y=0$, $\rho v=0$ SINCE $v=0$
- AT $y=\delta$, $\rho v=0$ SINCE $v=0$
- $\rho u|_{x+dx} = \rho u + \frac{\partial(\rho u)}{\partial x}dx$

THEN THE LATTER BECOMES:

$$\int_0^{\delta}\left(\cancel{\rho u} + \frac{\partial(\rho u)}{\partial x}dx - \cancel{\rho u}\right)dy + 0 = 0$$

AFTER DIVIDING BY dx:

$$\int_0^{\delta} \frac{\partial(\rho u)}{\partial x}\,dy = 0$$

NOW, WE CAN MULTIPLY THE LATTER BY A CONSTANT u_∞ (THE FREESTREAM VELOCITY):

$$u_\infty \int_0^\delta \frac{\partial(\rho u)}{\partial x}\, dy = 0$$

SINCE u_∞ IS A CONSTANT, IT CAN BE MOVED INSIDE THE INTEGRAL:

$$\int_0^\delta u_\infty \frac{\partial(\rho u)}{\partial x}\, dy = 0$$

AND MOVED FURTHER INSIDE THE DERIVATIVE:

$$\int_0^\delta \frac{\partial(\rho u u_\infty)}{\partial x}\, dy = 0$$

BUT THE INTEGRAL OF A DERIVATIVE IS EQUAL TO THE DERIVATIVE OF AN INTEGRAL:

$$\boxed{\frac{d}{dx}\int_0^\delta \rho u u_\infty\, dy = 0} \quad \text{II}$$

THEN, WE CAN SUBTRACT EQ. I FROM EQ. II:

$$\boxed{\frac{d}{dx}\int_0^\delta \rho u (u_\infty - u)\, dy = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}} \quad \text{III}$$

BUT WE NEED TO FIND u AS A FUNCTION OF y IN ORDER TO INTEGRATE THE LATTER. WE CAN APPROXIMATE THE VELOCITY DISTRIBUTION BY FITTING A POLYNOMIAL THROUGH THE BOUNDARY CONDITIONS. INDEED, FOR A BOUNDARY LAYER, THE FOLLOWING HOLDS TRUE:

(a) AT $y = \delta$, $u = u_\infty$

(b) AT $y = \delta$, $\dfrac{\partial u}{\partial y} = 0$

(c) AT $y = 0$, $u = 0$

(d) AT $y = 0$, $\dfrac{\partial^2 u}{\partial y^2} = 0$

THE LATTER CONDITION (d) ORIGINATES FROM THE X-MOMENTUM EQUATION AT $y = 0$:

$$\cancelto{0}{\frac{\partial(\rho u)}{\partial t}} + \cancelto{0}{\frac{\partial(\rho u^2)}{\partial x}} + \cancelto{0}{\frac{\partial(\rho u v)}{\partial y}} = \mu \cancelto{0}{\frac{\partial^2 u}{\partial x^2}} + \mu \frac{\partial^2 u}{\partial y^2}$$

ASSUMING THAT u CAN BE WELL REPRESENTED BY THE FOLLOWING POLYNOMIAL:

$$u \simeq C_1 + C_2 y + C_3 y^2 + C_4 y^3$$

WE CAN FIND THE COEFFICIENTS C_1, C_2, C_3, C_4 BY IMPOSING THE 4 BOUNDARY CONDITIONS (a), (b), (c), AND (d).

THIS THEN YIELDS:

$$u = u_\infty \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \right]$$

WHICH WE CAN DIFFERENTIATE WITH RESPECT TO y AT $y=0$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} \simeq \frac{3 u_\infty}{2\delta}$$

REPLACE THE LATTER 2 EQS IN EQ. III:

$$\frac{d}{dx}\int_0^\delta \rho u_\infty^2 \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] dy \simeq \frac{3\mu u_\infty}{2\delta}$$

AFTER INTEGRATION, WE GET:

$$\frac{d}{dx}\left(\frac{39}{280}\rho u_\infty^2 \delta\right) \simeq \frac{3}{2}\frac{\mu u_\infty}{\delta}$$

SINCE ρu_∞^2 IS CONSTANT, WE CAN MOVE IT OUT OF THE DERIVATIVE:

$$\frac{d}{dx}\left(\frac{39}{280}\delta\right) = \frac{3}{2}\frac{\mu}{\rho u_\infty \delta}$$

THEN MULTIPLY BY δdx AND INTEGRATE:

$$\int \frac{39}{280}\delta \, d\delta = \int \frac{3}{2}\frac{\mu}{\rho u_\infty} dx$$

$$\frac{39}{280}\frac{\delta^2}{2} = \frac{3}{2}\frac{\mu x}{\rho u_\infty} + C_1$$

INTEG. CONST.

NOW, IMPOSE B.C. TO DETERMINE C_1:

AT $x=0$, $\delta=0$

THEN:

$$0 = 0 + C_1$$

AND THE BOUNDARY LAYER THICKNESS BECOMES:

$$\delta \simeq \sqrt{\frac{280}{13}\frac{\mu x}{\rho u_\infty}}$$

BUT RECALL THE DEFINITION OF THE REYNOLDS NUMBER:

$$Re_x \equiv \frac{\rho u_\infty x}{\mu}$$

THEN THE BOUNDARY LAYER THICKNESS CAN BE EXPRESSED AS:

$$\delta \simeq \sqrt{\frac{280}{13}\frac{x^2}{Re_x}}$$

OR:

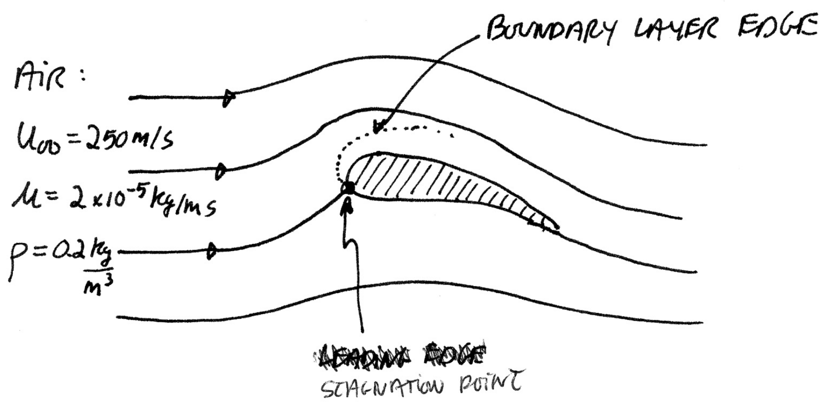
$$\boxed{\frac{\delta}{x} = \frac{4.64}{Re_x^{0.5}}} \quad (A)$$

FOLLOWING A SIMILAR DERIVATION AS JUST DONE BUT WITHOUT RESORTING TO A POLYNOMIAL APPROXIMATION TO THE VELOCITY, ONE CAN SHOW THAT THE EXACT SOLUTION TO THE LAMINAR BOUNDARY LAYER THICKNESS IS:

$$\boxed{\frac{\delta_{LAM}}{x} = \frac{5}{Re_x^{0.5}}}$$

GIVEN $\rho u_\infty/\mu$ AND NOTING THAT THE LATTER IS CONSTANT ALONG x, THE THICKNESS OF THE LAMINAR BOUNDARY LAYER CAN BE READILY FOUND.

FOR INSTANCE, CONSIDER AIR FLOWING OVER AN AIRCRAFT WING



WHAT IS BOUNDARY LAYER THICKNESS 10 cm DOWNSTREAM OF THE STAGNATION POINT?

FIRST FIND RE_x AT THAT POINT:

$$RE_{x=0.1m} = \frac{\rho u_\infty x}{\mu} = \frac{0.2\ kg/m^3 \times 250\ m/s \times 0.1m}{2\times 10^{-5}\ kg/ms}$$

$$RE_{x=0.1m} = 2.5 \times 10^5$$

THEN FIND δ_{LAM}:

$$\delta_{LAM} = \frac{5x}{RE_x^{0.5}} = \frac{5\times 0.1m}{(2.5\times10^5)^{0.5}} = 1\ mm$$

FOR A TURBULENT BODY LAYER, IT IS FOUND EMPIRICALLY THAT:

$$\boxed{\frac{\delta_{TURB}}{x} = \frac{0.381}{RE_x^{0.2}}}$$