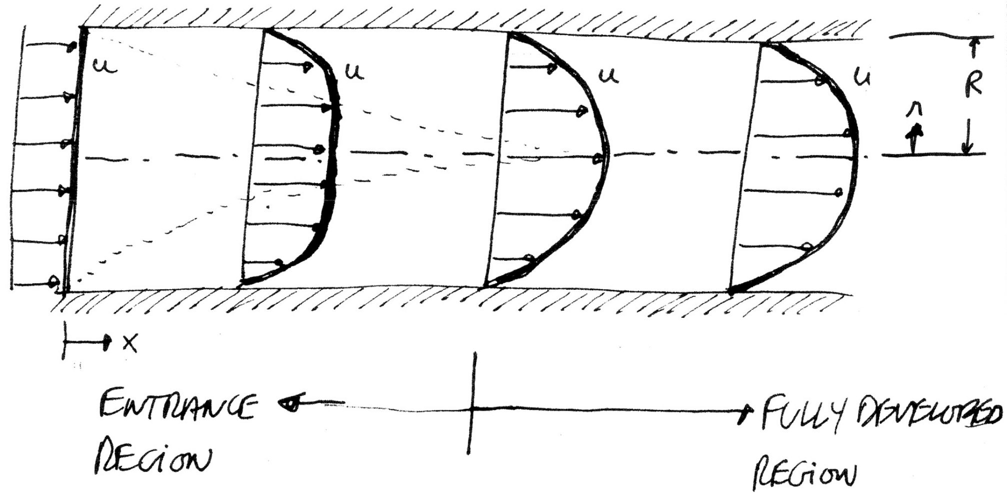
Heat Transfer Handout 3 — Fully-Developed Flow

FULLY DEVELOPED FLOW IN PIPE:



IN THE FULLY DEVELOPED REGION:

$$v = 0$$

THEN, FOR $v=0$ AND $\rho=$ CONSTANT, MASS CONSERVATION GIVES

$$\frac{\partial u}{\partial x} = 0$$

AND, FOR $v=0$, THE r-MOMENTUM EQUATION YIELDS:

$$\frac{\partial P}{\partial r} = 0$$

FOR $v=0$ AND $\partial u/\partial x = 0$, THE x-MOMENTUM EQUATION COLLAPSES TO:

$$\frac{\partial P}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \mu \frac{\partial u}{\partial r}\right)$$

MULTIPLY BY $r\,dr$ AND INTEGRATE:

$$\int r \frac{dP}{dx}\,dr = \int d\left(r \mu \frac{\partial u}{\partial r}\right)$$

SINCE dP/dx IS NOT A FUNCTION OF r SINCE $\partial P/\partial y = 0$:

$$\frac{dP}{dx}\frac{r^2}{2} = r\mu \frac{\partial u}{\partial r} + c_1$$

BECAUSE OF SYMMETRY WE CAN IMPOSE ONE B.C.:

AT $r = 0$, $\frac{\partial u}{\partial r} = 0$

THIS YIELDS

$$0 = 0 + c_1$$

REPLACING THE LATTER IN THE FORMER; AND SIMPLIFYING:

$$\frac{dP}{dx}\frac{r}{2} = \mu \frac{du}{dr}$$

NOW, MULTIPLY THE LATTER BY dr AND INTEGRATE:

$$\int \frac{dP}{dx}\frac{r}{2}\,dr = \int \mu\, du$$

$$\frac{dP}{dx}\frac{r^2}{4} = \mu u + C_2$$

IMPOSE 2nd B.C.

AT $r = R$, $u = 0$

$$\frac{dP}{dx}\frac{R^2}{4} = C_2$$

REPLACE LATTER IN FORMER:

$$\boxed{u = \frac{1}{\mu}\frac{dP}{dx}\left[\frac{r^2}{4} - \frac{R^2}{4}\right]} \qquad \mathrm{I}$$

KNOWING THE VELOCITY DISTRIBUTION, WE CAN FIND THE MASS FLOW RATE:

$$\dot{m} = \int_{r=0}^{r=R} \rho u\, dA$$

with $dA = 2\pi r\, dr$

SUBSTITUTE EQ. I IN LATTER:

$$\dot{m} = \int_{r=0}^{r=R} \frac{\rho}{\mu}\frac{dP}{dx}\left(\frac{r^2}{4} - \frac{R^2}{4}\right) 2\pi r \, dr$$

TAKE OUT THE CONSTANTS ρ, μ, 2π, dP/dx FROM INTEGRAL:

$$\dot{m} = \frac{\rho}{\mu}\frac{dP}{dx}\frac{2\pi}{4}\int_{r=0}^{r=R} (r^3 - rR^2)\,dr$$

$$\dot{m} = \frac{\rho}{\mu}\frac{\pi}{2}\frac{dP}{dx}\left(\frac{r^4}{4} - \frac{r^2R^2}{2}\right)_{r=0}^{r=R}$$

$$\dot{m} = -\frac{\rho}{\mu}\frac{\pi}{2}\frac{dP}{dx}\frac{R^4}{4}$$

$$\boxed{\dot{m} = -\frac{\rho}{\mu}\frac{\pi R^4}{8}\frac{dP}{dx}} \quad \underline{\mathrm{II}}$$

NOW WE WILL DEFINE THE MEAN BULK VELOCITY AS FOLLOWS:

$$\rho u_b A \equiv \dot{m}$$

WHICH YIELDS:

$$u_b \equiv \frac{\dot{m}}{\rho A}$$

SUBSTITUTE EQ. II IN THE LATTER:

$$u_b = -\frac{\cancel{\rho}\,\pi R^4}{\mu\, 8\, \cancel{\rho} A}\frac{dP}{dx}$$

AND NOTE THAT

$$A = \pi R^2$$

THEN WE GET:

$$\boxed{u_b = -\frac{R^2}{\mu\, 8}\frac{dP}{dx}} \quad \text{III}$$

NOW, WE CAN OBTAIN A NON-DIMENSIONAL VELOCITY PROFILE u/u_b BY DIVIDING EQ. I BY EQ. III:

$$\frac{u}{u_b} = \frac{\frac{1}{\cancel{\mu}}\frac{\cancel{dP}}{\cancel{dx}}\left[\frac{r^2}{4} - \frac{R^2}{4}\right]}{-\frac{R^2}{\cancel{\mu}8}\frac{\cancel{dP}}{\cancel{dx}}}$$

DIVIDE THE NUMERATOR AND DENOMINATOR BY R^2:

$$\boxed{\frac{u}{u_b} = \frac{\frac{r^2}{R^2} - 1}{-\frac{4}{8}} = 2\left(1 - \frac{r^2}{R^2}\right)}$$