2017 Numerical Analysis Final Exam

Numerical Analysis
Final Examination
Friday December 15, 2017
18:00 — 21:00

ANSWER ALL 6 PROBLEMS; ALL PROBLEMS HAVE EQUAL VALUE; NO NOTES OR BOOKS.

Question #1

Knowing that

$$|\epsilon_{n+1}|_{\text{Newton}} = \left|\frac{1}{2}\frac{f''(r)}{f'(r)}\right| |\epsilon_n|^2$$

and

$$|\epsilon_{n+1}|_{\text{secant}} = \left|\frac{1}{2}\frac{f''(r)}{f'(r)}\right|^{1/1.618} |\epsilon_n|^{1.618}$$

Prove that

$$\frac{|\epsilon_n|_{\text{Newton}}}{|\epsilon_n|_{\text{secant}}} = \left(\left|\frac{1}{2}\frac{f''(r)}{f'(r)}\right| |\epsilon_0|\right)^{(2^n - 1.618^n)}$$

Note: the latter should be proven fully without skipping steps or making an assumption/simplification.

Question #2

Using a second-order Runge-Kutta method, solve q at $t = 1$ for the RC circuit equation

$$R\frac{dq}{dt} + \frac{q}{C} = 0$$

with $RC = 3$, with the initial condition being $q_0 = 2$, with $\Delta t = 0.5$, and with the constraint $a = 0.5$. Do so in two different ways:

(a) By hand

(b) With a C code that starts as follows:

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <assert.h>
```

```
#define dt 0.5
#define tmax 1.0

double f(
```

Note: your algorithm should make use of the defined dt and tmax and work for any value of dt or tmax.

Question #3

Consider a real number stored with 5 bytes. Bit #1 is reserved for the sign, while bits #2 to #10 are reserved for the biased exponent, and bits #11 to #40 are related to the significand. Do the following:

(a) Find the minimum and maximum possible exponent p

(b) Find the smallest possible positive number

(c) Find the largest possible number

(d) Find the smallest possible positive subnormal number

Question #4

Consider the following data points:

x	$f(x)$
1	2
2	4
4	3

It is given that at $x = 1$, $f''' = 0$. Using a cubic spline, find the value of $f(x)$ at $x = 3$. Derive proper boundary conditions and perform basic verifications to ensure that your answer is correct.

Question #5

Consider the system of equations $AX = B$ with A equal to:

$$A = \begin{bmatrix} -2 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

and B equal to:

$$B = \begin{bmatrix} -1 \\ -7 \\ 3 \\ -6 \end{bmatrix}$$

Using partial pivoting *only when the pivot is zero*, find the lower and upper triangular matrices associated with matrix A. Outline *all the steps needed* to obtain the matrix L, the matrix U, and the permutation matrices. Also, indicate clearly how A can be written as a function of L, U, and the permutation matrices.

Question #6

You wish to create a new numerical integration method. To do so, you come up with the idea of evaluating the integral I_i by fitting a 2nd degree polynomial of the form

$$P_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2$$

through 3 data points within the ith interval. For this purpose, do the following:

(a) Express the polynomial coefficients a_i, b_i, and c_i as a function of the data points $(x_i,\ f_i)$, $(x_{i+1/2},\ f_{i+1/2})$, $(x_{i+1},\ f_{i+1})$.

(b) Using the polynomial coefficients derived in (a), find an expression for I_i over the interval $x_i \le x \le x_{i+1}$ and simplify as much as possible.

Recall

The 2nd order Runge-Kutta scheme can be expressed as

$$\begin{aligned} k_1 &= \Delta t f(t_n,\ \phi_n) \\ k_2 &= \Delta t f(t_n + \alpha \Delta t,\ \phi_n + \beta k_1) \\ \phi_{n+1} &= \phi_n + a k_1 + b k_2 \end{aligned}$$

with the constraints

$$\begin{aligned} a + b &= 1 \\ b\alpha &= \frac{1}{2} \\ b\beta &= \frac{1}{2} \end{aligned}$$

A third order spline can be expressed as:

$$f_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

$$d_i = y_i$$

$$a_i = (b_{i+1} - b_i)/(3\Delta x_i) \quad \text{for } 1 \le i \le N - 1$$

$$c_i = \frac{\Delta y_i}{\Delta x_i} - b_i \Delta x_i - \left(\frac{b_{i+1} - b_i}{3}\right) \Delta x_i \quad \text{for } 1 \le i \le N - 1$$

$$\Delta x_{i-1} b_{i-1} + 2\left(\Delta x_i + \Delta x_{i-1}\right) b_i + \Delta x_i b_{i+1} = 3\left(\frac{\Delta y_i}{\Delta x_i} - \frac{\Delta y_{i-1}}{\Delta x_{i-1}}\right) \quad \text{for } 2 \le i \le N - 1$$