2018 Numerical Analysis Final Exam

Numerical Analysis
Final Examination
Friday December 21st, 2018
18:00 — 21:00

ANSWER ALL 6 PROBLEMS; ALL PROBLEMS HAVE EQUAL VALUE; NO NOTES OR BOOKS.

Question #1

Consider a real number stored with 5 bytes. Bit #1 is reserved for the sign, while bits #2 to #10 are reserved for the biased exponent, and bits #11 to #40 are related to the significand. Do the following:

(a) Find the minimum and maximum possible exponent p

(b) Find the smallest possible positive number

(c) Find the largest possible number

(d) Find the smallest possible positive subnormal number

Question #2

Consider the following data points:

x	$f(x)$
1	2
2	4
4	3

It is given that at $x = 1$, $f''' = 0$. Using a cubic spline, find the value of $f(x)$ at $x = 3$. Derive proper boundary conditions and perform basic verifications to ensure that your answer is correct.

Question #3

Consider the system of equations $AX = B$ with A equal to:

$$A = \begin{bmatrix} -2 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

and B equal to:

$$B = \begin{bmatrix} -1 \\ -7 \\ 3 \\ -6 \end{bmatrix}$$

Using partial pivoting *only when the pivot is zero*, find the lower and upper triangular matrices associated with matrix A. Outline *all the steps needed* to obtain the matrix L, the matrix U, and the permutation matrices. Also, indicate clearly how A can be written as a function of L, U, and the permutation matrices.

Question #4

The secant method (a.k.a. Newton-Raphson) can be written in the following form:

$$x_{n+1} = x_n - \frac{f_n}{(f_n')_{\text{secant}}}$$

where n is the iteration count, $f_n = f(x_n)$, and

$$(f_n')_{\text{secant}} = \frac{f_n - f_{n-1}}{x_n - x_{n-1}}$$

Do the following:

(a) Improve the secant method by finding an approximation for f' that is second order accurate. That is, derive using the Taylor series an expression for $(f_n')_{\text{secant2}}$ such that

$$(f_n')_{\text{secant2}} = f_n' + O(\Delta x^2)$$

(b) Consider the function $f = \sin(x)$ with x in radians. Write a C code that finds the root $f = 0$ with an absolute error on f not exceeding 10^{-9} for the initial condition $x_0 = 2.8$ using the second-order secant method derived in (a). The C code should start as follows:

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <assert.h>

double f(double x){
  double ret;
  ret=sin(x);
  return(ret);
}
```

```
int main(void){
```

EDIT Numerical_Analysis_F2018Q4.c

Question #5

Use Taylor series to derive a numerical differentiation method for the ODE

$$\frac{d\phi}{dt} = f(\phi, t)$$

when given $f(\phi, t)$ and $g(\phi, t) = d^2\phi/dt^2$ and $h(\phi, t) = d^3\phi/dt^3$. Note that f and g and h are given expressions and are thus free of numerical error. Specifically, do the following:

(a) First derive a method that is third-order accurate. You should also prove that the global error on $\phi_N - \phi_0$ is $O(\Delta t^3)$.

(b) Then derive a method that is fourth-order accurate. You should also prove that the global error on $\phi_N - \phi_0$ is $O(\Delta t^4)$.

Question #6

You fit a curve using the method of least squares through the following data points:

	x	y
1	0	$9/2$
2	$\pi/4$	$\sqrt{2}$
3	$\pi/2$	?

and with the function:

$$y = c_1 \sin(x) + c_2 \cos(x)$$

with x in radians. If the method of least squares yields $c_1 = 2$ and $c_2 = 3$, find y_3.

Reminder

Least square fit of a combination of functions:

$$(A^T A)C = A^T Y$$

Equations for inner nodes within cubic splines:

$$f_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

$$d_i = y_i$$

$$a_i = (b_{i+1} - b_i)/(3\Delta x_i) \quad \text{for } 1 \le i \le N - 1$$

$$c_i = \frac{\Delta y_i}{\Delta x_i} - b_i \Delta x_i - \left(\frac{b_{i+1} - b_i}{3}\right)\Delta x_i \quad \text{for } 1 \le i \le N - 1$$

$$\Delta x_{i-1}b_{i-1} + 2\left(\Delta x_i + \Delta x_{i-1}\right)b_i + \Delta x_i b_{i+1} = 3\left(\frac{\Delta y_i}{\Delta x_i} - \frac{\Delta y_{i-1}}{\Delta x_{i-1}}\right) \quad \text{for } 2 \leq i \leq N - 1$$