Computational Aerodynamics Assignment 4 — Flux Jacobian and Eigenvalues

Question #1

Starting from the Euler equations:

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix} \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + P \\ \rho u H \end{bmatrix}$$

Show that the vectors U and F can be expressed as:

$$U = \begin{bmatrix} \rho \\ \rho u \\ \frac{1}{\gamma-1}\rho d + \frac{1}{2}\rho u^2 \end{bmatrix} \text{ and } F = \begin{bmatrix} \rho u \\ \rho u^2 + \rho d \\ \frac{\gamma}{\gamma-1}\rho u d + \frac{1}{2}\rho u^3 \end{bmatrix}$$

with $d \equiv RT$ and $\gamma \equiv c_p/c_v$. Outline clearly your assumptions.

Question #2

Starting from F and U found in Question #1, prove that the flux Jacobian $A \equiv \partial F/\partial U$ is equal to:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma - 1 \\ \frac{\gamma}{1-\gamma}ud + \frac{\gamma-2}{2}u^3 & \frac{\gamma}{\gamma-1}d + \frac{3-2\gamma}{2}u^2 & \gamma u \end{bmatrix}$$

Note: you only need to prove the terms on the third row of A. Do not derive the terms on the first and second rows.

Question #3

Starting from the flux Jacobian obtained in the previous question, show that u is a valid wavespeed of the Euler equations.

Question #4

Consider the following system of equations:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

with

$$U = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho u \\ \rho E \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} \rho_1 u \\ \rho_2 u \\ \rho u^2 + P \\ \rho u H \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

with

$$E = \frac{\rho_1}{\rho} c_{v1} T + \frac{\rho_2}{\rho} c_{v2} T + \frac{u^2}{2}$$

$$H = \frac{\rho_1}{\rho} c_{p1} T + \frac{\rho_2}{\rho} c_{p2} T + \frac{u^2}{2}$$

$$P = (\rho_1 R_1 + \rho_2 R_2) T$$

$$\rho = \rho_1 + \rho_2$$

and with c_{v1}, c_{v2}, c_{p1}, c_{p2}, R_1, R_2 some constants. Using the method of your choice find the following elements within the flux Jacobian:

(a) Find $\partial F_1 / \partial U_1$

(b) Find $\partial F_3 / \partial U_4$

Hint: $\rho H = \rho E + P$.

Question #5

Consider the following system of equations:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

with

$$U = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho u \\ \rho E \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} \rho_1 u \\ \rho_2 u \\ \rho u^2 + P \\ \rho u H \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

with

$$E = \frac{\rho_1}{\rho} e_1 + \frac{\rho_2}{\rho} e_2 + \frac{u^2}{2}$$

$$P = (\rho_1 R_1 + \rho_2 R_2) T$$

$$\rho = \rho_1 + \rho_2$$

$$H = E + \frac{P}{\rho}$$

$$e_1 = \xi_1 + \xi_2 T + \xi_3 T^2 + \xi_4 T^3 + \xi_5 T^4$$

$$e_2 = \xi_6 T$$

and with ξ_1, ξ_2, ξ_3, ξ_4, ξ_5, ξ_6, R_1, R_2 some constants. Find $\partial F_3 / \partial U_4$ within the flux Jacobian.

Answers

4. $u\left(1-\frac{\rho_1}{\rho}\right)$, $\frac{\rho_1 R_1+\rho_2 R_2}{c_{v1}\rho_1+c_{v2}\rho_2}$.
5. $\frac{\rho_1 R_1+\rho_2 R_2}{\rho_1(\xi_2+2\xi_3 T+3\xi_4 T^2+4\xi_5 T^3)+\rho_2\xi_6}$

Due on Tuesday April 23rd at 16:30. Do Questions #4 and #5 only.