

# Computational Aerodynamics

## Assignment 4 — Flux Jacobian and Eigenvalues

### Question #1

Starting from the Euler equations:

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix} \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + P \\ \rho u H \end{bmatrix}$$

Show that the vectors  $U$  and  $F$  can be expressed as:

$$U = \begin{bmatrix} \rho \\ \rho u \\ \frac{1}{\gamma-1}\rho d + \frac{1}{2}\rho u^2 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + \rho d \\ \frac{\gamma}{\gamma-1}\rho u d + \frac{1}{2}\rho u^3 \end{bmatrix}$$

with  $d \equiv RT$  and  $\gamma \equiv c_p/c_v$ . Outline clearly your assumptions.

### Question #2

Starting from  $F$  and  $U$  found in Question #1, prove that the flux Jacobian  $A \equiv \partial F/\partial U$  is equal to:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma-1 \\ \frac{\gamma}{1-\gamma}ud + \frac{\gamma-2}{2}u^3 & \frac{\gamma}{\gamma-1}d + \frac{3-2\gamma}{2}u^2 & \gamma u \end{bmatrix}$$

Note: you only need to prove the terms on the third row of  $A$ . Do not derive the terms on the first and second rows.

### Question #3

Starting from the flux Jacobian obtained in the previous question, show that  $u$  is a valid wavespeed of the Euler equations.

### Question #4

Consider the following system of equations:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

with

$$U = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho u \\ \rho E \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} \rho_1 u \\ \rho_2 u \\ \rho u^2 + P \\ \rho u H \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

with

$$E = \frac{\rho_1}{\rho}c_{v1}T + \frac{\rho_2}{\rho}c_{v2}T + \frac{u^2}{2}$$

$$H = \frac{\rho_1}{\rho}c_{p1}T + \frac{\rho_2}{\rho}c_{p2}T + \frac{u^2}{2}$$

$$P = (\rho_1 R_1 + \rho_2 R_2)T$$

$$\rho = \rho_1 + \rho_2$$

and with  $c_{v1}$ ,  $c_{v2}$ ,  $c_{p1}$ ,  $c_{p2}$ ,  $R_1$ ,  $R_2$  some constants. Using the method of your choice find the following elements within the flux Jacobian:

(a) Find  $\partial F_1/\partial U_1$

(b) Find  $\partial F_3/\partial U_4$

Hint:  $\rho H = \rho E + P$ .

### Question #5

Consider the following system of equations:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

with

$$U = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho u \\ \rho E \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} \rho_1 u \\ \rho_2 u \\ \rho u^2 + P \\ \rho u H \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

with

$$E = \frac{\rho_1}{\rho}e_1 + \frac{\rho_2}{\rho}e_2 + \frac{u^2}{2}$$

$$P = (\rho_1 R_1 + \rho_2 R_2)T$$

$$\rho = \rho_1 + \rho_2$$

$$H = E + \frac{P}{\rho}$$

$$e_1 = \xi_1 + \xi_2 T + \xi_3 T^2 + \xi_4 T^3 + \xi_5 T^4$$

$$e_2 = \xi_6 T$$

and with  $\xi_1$ ,  $\xi_2$ ,  $\xi_3$ ,  $\xi_4$ ,  $\xi_5$ ,  $\xi_6$ ,  $R_1$ ,  $R_2$  some constants. Find  $\partial F_3/\partial U_4$  within the

flux Jacobian.

**Answers**

4.  $u \left( 1 - \frac{\rho_1}{\rho} \right), \frac{\rho_1 R_1 + \rho_2 R_2}{c_{v1} \rho_1 + c_{v2} \rho_2}.$

5.  $\frac{\rho_1 R_1 + \rho_2 R_2}{\rho_1 (\xi_2 + 2\xi_3 T + 3\xi_4 T^2 + 4\xi_5 T^3) + \rho_2 \xi_6}$

**Due on Tuesday April 23rd at 16:30. Do Questions #4 and #5 only.**