Fundamentals of Fluid Mechanics B
Assignment 10 — Vorticity Dynamics

Instructions

Write your solutions in single column format, with one statement following another vertically. Write your solutions neatly so that they are easy to read and verify. Don't write one line with two equal signs. Highlight your answers using a box. Failure to do this will result in a lower score and fewer comments on my part.

Question #1

Derive Kelvin's circulation theorem. Specifically, do the following:

(a) Prove that

$$\frac{d(d\xi_i)}{dt} = dv_i$$

(b) Prove that

$$\oint_C d\left(\frac{v_i^2}{2}\right) = 0$$

(c) Prove that

$$\oint_C \frac{\partial P}{\partial x_i} d\xi_i = 0$$

(d) Using (a), (b), (c), and starting from the definition of the circulation:

$$\Gamma \equiv \oint_C \vec{v} \cdot d\vec{\xi}$$

Prove that the following is correct:

$$\frac{d\Gamma}{dt} = \sum_{i=1}^{3} \oint_C \left(\frac{\mu}{\rho}(\vec{\nabla} \cdot \vec{\nabla})v_i\right) d\xi_i$$

Question #2

Starting from the vorticity equation:

$$\frac{d\vec{\omega}}{dt} = (\vec{\omega} \cdot \vec{\nabla})\vec{v} + \frac{\mu}{\rho}(\vec{\nabla} \cdot \vec{\nabla})\vec{\omega}$$

Do the following:

(a) Show that the first term on the RHS can be rewritten as:

$$(\vec{\omega} \cdot \vec{\nabla})\vec{v} = \frac{\omega}{L}\frac{d}{dt}(L\vec{n})$$

with L the length of the vortex line, $\vec{n}$ a unit vector pointing in the direction of the vorticity vector, and ω the magnitude of the vorticity.

(b) Explain why the term $(\vec{\omega} \cdot \vec{\nabla})\vec{v}$ is called vortex stretching.

Due on Wednesday May 6th 6:00 pm but submissions accepted until Sunday May 10th 11:00 am. Do all problems.