Fundamentals of Fluid Mechanics B
Handout 1 — Mangler's Transform

MANGLER'S TRANSFORM

MANGLER (1945) SHOWED THAT WE CAN TRANSFORM THE AXISYMMETRIC EQUATIONS FOR A BOUNDARY LAYER

$$\frac{\partial}{\partial s}\left(R v_s\right) + R\,\frac{\partial v_n}{\partial n} = 0$$

$$\rho v_s \frac{\partial v_s}{\partial s} + \rho v_n \frac{\partial v_s}{\partial n} = \mu \frac{\partial^2 v_s}{\partial n^2}$$

to THE FOLLOWING "PLANAR" FORM:

$$\frac{\partial \hat{v}_s}{\partial \hat{s}} + \frac{\partial \hat{v}_n}{\partial \hat{n}} = 0$$

$$\rho \hat{v}_s \frac{\partial \hat{v}_s}{\partial \hat{s}} + \rho \hat{v}_n \frac{\partial \hat{v}_s}{\partial \hat{n}} = \mu \frac{\partial^2 \hat{v}_s}{\partial \hat{n}^2}$$

THROUGH THE FOLLOWING TRANSFORM:

$$\boxed{\begin{aligned} \hat{s} &\equiv \int_0^s \frac{R^2}{L^2}\,ds & \qquad \hat{v}_s &= v_s \\ \hat{n} &\equiv \frac{R}{L}\,n & \qquad \hat{v}_n &= \frac{L}{R}\left(v_n + \frac{n}{R} v_s \frac{dR}{ds}\right) \end{aligned}}$$

WITH L A CERTAIN LENGTH SCALE (CONSTANT THROUGH DOMAIN)
(A CHARACTERISTIC LENGTH OF BODY)

TAKE $\partial/\partial s$ ON BOTH SIDES OF LATTER:

$$\frac{\partial}{\partial s}(\hat{n}) = \frac{\partial}{\partial s}\left(\frac{R}{L} n\right)$$

BUT $R = R(s)$ AND n DOES NOT DEPEND ON s.

THUS:

$$\boxed{\frac{\partial \hat{n}}{\partial s} = \frac{n}{L}\frac{dR}{ds}}$$

ALSO TAKE $\partial/\partial s$ ON BOTH SIDES OF $\hat{s}$ DEFINITION

$$\frac{\partial}{\partial s}(\hat{s}) = \frac{\partial}{\partial s}\left(\int_0^s \frac{R^2}{L^2}\, ds\right)$$

$\therefore$

$$\boxed{\frac{\partial \hat{s}}{\partial s} = \frac{R^2}{L^2}}$$

WE CAN FIND THE OPERATOR $\partial/\partial s$ THROUGH THE CHAIN RULE

$$\frac{\partial}{\partial s}(\cdot) = \frac{\partial \hat{s}}{\partial s}\frac{\partial}{\partial \hat{s}}(\cdot) + \frac{\partial \hat{n}}{\partial s}\frac{\partial}{\partial \hat{n}}(\cdot)$$

SUBSTITUTE:

$$\boxed{\frac{\partial(\cdot)}{\partial s} = \frac{R^2}{L^2}\frac{\partial(\cdot)}{\partial \hat{s}} + \frac{n}{L}\frac{dR}{ds}\frac{\partial(\cdot)}{\partial \hat{n}}}$$

WE CAN FIND THE OPERATOR $\frac{\partial}{\partial n}$ THROUGH THE CHAIN RULE:

$$\frac{\partial(\cdot)}{\partial n} = \frac{\partial \hat{n}}{\partial n}\frac{\partial}{\partial \hat{n}}(\cdot) + \frac{\partial \hat{s}}{\partial n}\frac{\partial(\cdot)}{\partial \hat{s}}$$

BUT

$$\frac{\partial \hat{n}}{\partial n} = \frac{\partial}{\partial n}\left(\frac{R}{L}n\right)$$

BECAUSE $R = R(s)$ and L is CONSTANT:

$$\frac{\partial \hat{n}}{\partial n} = \frac{R}{L}$$

ALSO

$$\frac{\partial \hat{s}}{\partial n} = \frac{\partial}{\partial n}\left(\int_0^s \frac{R^2}{L^2}\, ds\right)$$

But the integral does not depend on n.

SO

$$\frac{\partial \hat{s}}{\partial n} = 0$$

$\therefore$

$$\boxed{\frac{\partial}{\partial n}(\cdot) = \frac{R}{L}\frac{\partial(\cdot)}{\partial \hat{n}} + 0}$$

FURTHER FIND v_s AND v_n :

$$\boxed{v_s = \hat{v}_s}$$

AND

$$\frac{R\hat{v}_n}{L} = v_n + \frac{n}{R} v_s \frac{dR}{ds}$$

$$\boxed{v_n = \frac{R\hat{v}_n}{L} - \frac{n}{R} \hat{v}_s \frac{dR}{ds}}$$

NOW RECALL AXI BDRY LAYER EQ :

$$\frac{\partial}{\partial s}(Rv_s) + R\frac{\partial v_n}{\partial n} = 0$$

EXPAND

$$v_s \frac{dR}{ds} + R\frac{\partial v_s}{\partial s} + R\frac{\partial v_n}{\partial n} = 0$$

SUBST THE OPERATORS $\partial/\partial s$ AND $\partial/\partial n$:

$$v_s \frac{dR}{ds} + R\left(\frac{R^2}{L^2}\frac{\partial v_s}{\partial \hat{s}} + \frac{n}{L}\frac{dR}{ds}\frac{\partial v_s}{\partial \hat{n}}\right) + R\frac{R}{L}\frac{\partial v_n}{\partial \hat{n}} = 0$$

SUBSTITUTE

$$v_s = \hat{v}_s \quad \text{AND} \quad v_n = \frac{R\hat{v}_n}{L} - \frac{n}{R}\hat{v}_s\frac{dR}{ds}$$

$\therefore$

$$\hat{v}_s\frac{dR}{ds} + R\left(\frac{R^2}{L^2}\frac{\partial \hat{v}_s}{\partial \hat{s}} + \frac{n}{L}\frac{dR}{ds}\frac{\partial \hat{v}_s}{\partial \hat{n}}\right) + \frac{R^2}{L}\frac{\partial}{\partial \hat{n}}\left(\frac{R\hat{v}_n}{L} - \frac{n}{R}\hat{v}_s\frac{dR}{ds}\right) =$$

RECALL:

$$\hat{n} = \frac{R}{L}n$$

SO

$$n = \frac{L\hat{n}}{R}$$

SUBSTITUTE:

$$\hat{v}_s\frac{dR}{ds} + \frac{R^3}{L^2}\frac{\partial \hat{v}_s}{\partial \hat{s}} + \frac{\cancel{R}\cancel{L}}{\cancel{L}\cancel{R}}\hat{n}\frac{dR}{ds}\frac{\partial \hat{v}_s}{\partial \hat{n}} + \frac{R^2}{L}\frac{\partial}{\partial \hat{n}}\left(\frac{R\hat{v}_n}{L} - \frac{L\hat{n}}{R^2}\hat{v}_s\frac{dR}{ds}\right) = 0$$

SIMPLIFY:

$$\cancel{\hat{v}_s\frac{dR}{ds}} + \frac{R^3}{L^2}\frac{\partial \hat{v}_s}{\partial \hat{s}} + \cancel{\hat{n}\frac{dR}{ds}\frac{\partial \hat{v}_s}{\partial \hat{n}}} + \frac{R^3}{L^2}\frac{\partial \hat{v}_n}{\partial \hat{n}} - \cancel{\frac{R^2 L}{L R^2}\hat{v}_s\frac{dR}{ds}} - \cancel{\frac{R^2 L\hat{n}}{L R^2}\frac{\partial \hat{v}_s}{\partial \hat{n}}\frac{dR}{ds}} = 0$$

THIS YIELDS:

$$\frac{R^3}{L^2}\frac{\partial \hat{v}_s}{\partial \hat{s}} + \frac{R^3}{L^2}\frac{\partial \hat{v}_n}{\partial \hat{n}} = 0$$

MULTIPLY BY L^2/R^3:

$$\boxed{\frac{\partial \hat{v}_s}{\partial \hat{s}} + \frac{\partial \hat{v}_n}{\partial \hat{n}} = 0}$$

THIS SHOWS THAT THE MANGLER'S TRANSFORM REDUCES THE AXISYMMETRIC BDRY LAYER EQUATIONS FOR MASS CONS. TO PLANAR FORM.

IS THIS ALSO TRUE FOR THE MOMENTUM EQ.?

START FROM THE S-MOMENTUM EQ.

$$\rho v_s \frac{\partial v_s}{\partial s} + \rho v_n \frac{\partial v_s}{\partial n} = \mu \frac{\partial^2 v_s}{\partial n^2}$$

BUT

$$\frac{\partial}{\partial s}(\cdot) = \frac{R^2}{L^2}\frac{\partial(\cdot)}{\partial \hat{s}} + \frac{n}{L}\frac{dR}{ds}\frac{\partial(\cdot)}{\partial \hat{n}}$$

$$\frac{\partial}{\partial n}(\cdot) = \frac{R}{L}\frac{\partial(\cdot)}{\partial \hat{n}}$$

$$v_s = \hat{v}_s$$

SUBSTITUTE:

$$\rho \hat{v}_s \left(\frac{R^2}{L^2} \frac{\partial \hat{v}_s}{\partial \hat{s}} + \frac{n}{L} \frac{dR}{ds} \frac{\partial \hat{v}_s}{\partial \hat{n}} \right) + \rho v_n \frac{R}{L} \frac{\partial \hat{v}_s}{\partial \hat{n}} = \mu \frac{R^2}{L^2} \frac{\partial^2 \hat{v}_s}{\partial \hat{n}^2}$$

BUT

$$v_n = \frac{R \hat{v}_n}{L} - \frac{n}{R} \hat{v}_s \frac{dR}{ds}$$

$\therefore$

$$\rho \hat{v}_s \left(\frac{R^2}{L^2} \frac{\partial \hat{v}_s}{\partial \hat{s}} + \frac{n}{L} \frac{dR}{ds} \frac{\partial \hat{v}_s}{\partial \hat{n}} \right) + \frac{\rho R}{L} \left(\frac{R \hat{v}_n}{L} - \frac{n}{R} \hat{v}_s \frac{dR}{ds} \right) \frac{\partial \hat{v}_s}{\partial \hat{n}} = \frac{\mu R^2}{L^2} \frac{\partial^2 \hat{v}_s}{\partial \hat{n}^2}$$

EXPAND:

$$\rho \hat{v}_s \frac{R^2}{L^2} \frac{\partial \hat{v}_s}{\partial \hat{s}} + \cancel{\rho \hat{v}_s \frac{n}{L} \frac{dR}{ds} \frac{\partial \hat{v}_s}{\partial \hat{n}}} + \frac{\rho R^2}{L^2} \hat{v}_n \frac{\partial \hat{v}_s}{\partial \hat{n}} - \cancel{\frac{\rho n}{L} \hat{v}_s \frac{dR}{ds} \frac{\partial \hat{v}_s}{\partial \hat{n}}} = \frac{\mu R^2}{L^2} \frac{\partial^2 \hat{v}_s}{\partial \hat{n}^2}$$

SIMPLIFY AND DIVIDE BY R^2/L^2:

$$\boxed{\rho \hat{v}_s \frac{\partial \hat{v}_s}{\partial \hat{s}} + \rho \hat{v}_n \frac{\partial \hat{v}_s}{\partial \hat{n}} = \mu \frac{\partial^2 \hat{v}_s}{\partial \hat{n}^2}}$$

AGAIN WE SEE THAT THE MANGLER'S TRANSFORM REDUCES THE AXISYMMETRIC TRANSPORT EQUATION TO "PLANAR" FORM.